Comments on efficiency and the Cramér-Rao lower bound (23 march 2020)

Relative efficiency

We have seen so far that *unbiasedness* is a desirable property, in principle; it can be thought of as the absence of systematic deviation of our estimator $\hat{\theta}$ from the target θ. The formal condition is $E[\hat{\theta}] = \theta$.

We have seen further that *consistency*, requiring $\hat{\theta} \xrightarrow{p} \theta$ is really the minimum we are willing to settle for.

The questions arises now that there are many estimators. We might be tempted to search for "the best", but this is futile (slide **Efficiency (I)**): there is no such thing as an estimator which is always better than any other.

This leads us to ask a less ambitious question: among a restricted class of estimators, such as the class of unbiased estimators, is there one which according to the criterion of MSE (minimum square error) is best? Remember that $MSE = \sigma^2 + (bias)^2$, so comparing MSE among unbiased estimators amounts to the comparison of variances (for the bias is zero). Comparison among unbiased estimators using a MSE criterion boils down to comparison of variances, and the ratio of variances of any two estimators is called their *relative efficiency* (slide **Efficiency (II)**; the next two slides supply examples on how to compute it).

Efficiency

Relative efficiency is a useful concept to compare a pair of unbiased estimators but we are out in search of "the best" among unbiased estimators. It makes sense then to ask ourselves what is the minimum variance that we can achieve (for a given sample size) with an unbiased estimator. Any estimator which reaches that minimum variance would qualify as "best" or "as good as the best" in the class of unbiased: we will say the it is *efficient.*

But what is that minimum variance? There is a remarkable result, almost simultaneously discovered in the last years of World War II (and published shortly after) by the Swedish actuary Harald Cramér and Indian statistician C.R. Rao (slide **The Cramér-Rao lower bound**). In rather general conditions (so called *regularity conditions*), the minimum variance of an unbiased estimator can be computed rather easily. The proof of the CR lower bound is not difficult: you can find it in a number of books, including Garthwaite et al. (1995), Cramér (1960) and a number of others (but we shall not present it nor require it in an exam).

The importance of this is quite clear: we can compute the least possible variance of an unbiased estimator *without knowing such estimator.* Once we find one that attains the CR lower bound, there is no point in searching more: nothing better exists. Knowing before we start what is the best we can attain, we know when we can stop searching.

An unbiased estimator that reaches the minimum attainable variance prescribed by the CR lower bound is called *efficient.*

Regularity conditions and computation of the CR bound

Slide **What are those regularity conditions?** gives a simplified view of the requirements for the CR result to hold. Basically, the density has to be smooth and "differentiable enough" with respect to the parameter; and, most important (as it is often the condition that fails), the range of the distribution must not depend on the parameter.

> For instance, in a Poisson distribution the variable can take values $0, 1, 2, 3, \ldots$, whichever the value of λ. Compare with the (non-regular case) of a $U(0, \theta)$ where the values X can take change with the value of θ.

Slide **A trick to compute the Cramér-Rao bound** shows a useful relationship. Sometimes the expectation of the squared first derivative of the log likelihood is difficult to compute and the expectation of the second derivative is easy. The relationship among both enables us to compute whichever is simpler.

Finally, you have some examples of computation of the CR in common situations.

References

H. Cramér. *Métodos Matemáticos de Estadística.* Ed. Aguilar, Madrid, 1970 edition, 1960.

P. H. Garthwaite, I. T. Jolliffe, and B. Jones. *Statistical Inference.* Prentice Hall, London, 1995.